Mass Properties of Triangulated Solids and Their Derivatives

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Abstract

This supplemental material describes the computation of mass properties of triangulated solids and their derivatives w.r.t. surface vertices. We start by briefly reviewing the volume integrals for mass, center of mass, and moment of inertia. Thereafter, we reduce the volume to surface integrals using the Divergence theorem, resulting in analytical expressions for a volume bounded by a triangulated surface. We then discuss derivatives of these analytical surface integrals w.r.t. vertices. We provide pseudo code for both mass properties and their derivatives for the reader's convenience. The resulting routines serve as fundamental building blocks for optimizing moment of inertia for spinnable objects.

1 Mass Properties

For a model $\mathcal{M}$, the mass properties are mass M, center of mass $\mathbf{c}$, and the 3×3 symmetric moment of inertia tensor $\mathbf{I}$. Assume that the surface of $\mathcal{M}$ encloses a region $\Omega \in \mathbb{R}^3$ that corresponds to a solid object with constant density ρ. We express the above quantities by collecting the monomials t of degree ≤ 2 in the 10-vector

$$\mathbf{t} = \left[\begin{array}{c|ccc|ccc|ccc} 1 & x & y & z & xy & yz & xz & x^2 & y^2 & z^2 \end{array}\right],$$

then taking the integrals over Ω:

$$\mathbf{s}_\Omega(\rho) = [s_1, s_x, s_y, s_z, s_{xy}, s_{yz}, s_{xz}, s_{x^2}, s_{y^2}, s_{z^2}]^T,$$
$$\text{where } \; s_t = \rho \int_\Omega t \, dV, \quad \text{e.g.,} \quad s_{xy} = \rho \int_\Omega xy \, dV.$$

We obtain the following expressions for the mass and center of mass:

$$M = s_1 \;\text{ and }\; \mathbf{c} = \frac{1}{M}\left[s_x, s_y, s_z\right]^T,$$

and $\mathcal{M}$'s inertia tensor:

$$\mathbf{I} = \begin{bmatrix} s_{y^2}+s_{z^2} & -s_{xy} & -s_{xz} \\ -s_{xy} & s_{x^2}+s_{z^2} & -s_{yz} \\ -s_{xz} & -s_{yz} & s_{x^2}+s_{y^2} \end{bmatrix}.$$

2 From Volume to Surface Integrals

Next, we reduce the volume to surface integrals. To this end, we identify a vector field T for each component t in the 10-vector $\mathbf{t}$ s.t. $\nabla \cdot T = t$, resulting in

$$\mathbf{T} = \left[\begin{array}{c|ccc|ccc|ccc} x & \frac{x^2}{2} & 0 & 0 & \frac{x^2 y}{2} & 0 & 0 & \frac{x^3}{3} & 0 & 0 \\ 0 & 0 & \frac{y^2}{2} & 0 & 0 & \frac{y^2 z}{2} & 0 & 0 & \frac{y^3}{3} & 0 \\ 0 & 0 & 0 & \frac{z^2}{2} & 0 & 0 & \frac{xz^2}{2} & 0 & 0 & \frac{z^3}{3} \end{array}\right].$$

We can then apply the Divergence Theorem to reduce our volume integrals $\mathbf{s}_\Omega$ over the region Ω to surface integrals over $\partial\Omega$

$$\mathbf{s}_\Omega^T(\rho) = \rho \int_\Omega \mathbf{t} \, dV = \rho \int_\Omega \nabla^T \mathbf{T} \, dV = \rho \int_{\partial\Omega} \mathbf{n}^T \mathbf{T} dS$$

with the unit normal $\mathbf{n}$ at point $[x, y, z]^T$.

If $\partial\Omega$ consists of a union of consistently oriented triangles $\mathcal{T}$, our integrals can be split into a sum of integrals over the individual faces

$$\mathbf{s}_\Omega(\rho) = \sum_{i\in\mathcal{T}} \mathbf{s}_i \;\text{ with }\; \mathbf{s}_i = \rho \int_i \mathbf{T}^T \mathbf{n}_i \, dS$$

with the unit triangle normals $\mathbf{n}_i$. In the following, we are omitting the triangle index i.

Because $\mathbf{T}$ has a single non-zero component T per column, the remaining integrals are of the form $\mathbf{e}_k \cdot \mathbf{n} \int T \, dS$ where $\mathbf{e}_k$ is a column of the unit matrix $\mathbf{E} = [\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z]$ and k denoting the row index of the non-zero T.

If we assume consistently oriented triangles whose vertices $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are ordered counterclockwise, the set of triangle points $[x, y, z]^T$ is

$$\mathbf{a} + \alpha\mathbf{u} + \beta\mathbf{v}$$

with limits $\alpha, \beta \in [0, 1]$ and $\alpha + \beta < 1$, and the face normal $\mathbf{n}$ and surface element dS are

$$\frac{\mathbf{u}\times\mathbf{v}}{|\mathbf{u}\times\mathbf{v}|} \;\text{ and }\; |\mathbf{u}\times\mathbf{v}| d\alpha d\beta,$$

respectively, where $\mathbf{u} = \mathbf{b} - \mathbf{a}$ and $\mathbf{v} = \mathbf{c} - \mathbf{a}$ denote the two triangle side vectors. With the above parameterization, our surface integrals become $\mathbf{e}_k \cdot \mathbf{n} \int_0^1 \int_0^{1-\beta} T d\alpha \, d\beta$ where $\mathbf{n}$ denotes the *un-normalized* normal $\mathbf{u}\times\mathbf{v}$ and T is expressed with the two triangle parameters α and β.

By calculating the analytical integrals (see, e.g., [Eberly 2003]), we can derive Alg. 1 (see next page) where $*$ denotes component-wise multiplication, $\mathrel{+}=$ and $\mathrel{*}=$ addition and multiplication assignment, respectively, and $\bar{\mathbf{v}}$ equals $[v_y, v_z, v_x]^T$ for a vector $\mathbf{v} = [v_x, v_y, v_z]^T$. Further, $\mathbf{o}_{10}$ denotes the zero 10-vector.

3 Taking Derivatives

For the derivatives w.r.t. the column vector $\mathbf{V}$ collecting all n surface vertices (Alg. 2), we also define the operators $[\mathbf{v}]_\times$ (conversion to skew-symmetric matrix) and $[\mathbf{v}]_*$:

$$[\mathbf{v}]_\times = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}, \; [\mathbf{v}]_* = \begin{bmatrix} 0 & v_x & 0 \\ 0 & 0 & v_y \\ v_z & 0 & 0 \end{bmatrix}.$$

We use $\mathbf{d}_{i,\mathbf{v}}$ to refer to the diagonal of the derivative $\frac{\partial \mathbf{h}_i}{\partial \mathbf{v}}$ and $\mathbf{H}_i$ to refer to matrix $[\mathbf{h}_i, \mathbf{h}_i, \mathbf{h}_i]$. Because $\mathbf{d}_{1,\mathbf{a}} = \mathbf{d}_{1,\mathbf{b}} = \mathbf{d}_{1,\mathbf{c}} = [1, 1, 1]^T$ and $\mathbf{d}_{4,\mathbf{a}} = \mathbf{h}_5, \; \mathbf{d}_{4,\mathbf{b}} = \mathbf{h}_6, \; \mathbf{d}_{4,\mathbf{c}} = \mathbf{h}_7$, we directly use the respective right-hand sides in Alg. 2. Further, we use curly brackets to group similar expressions. E.g., we group expressions $\frac{\partial \mathbf{w}}{\partial \mathbf{v}_1} = \mathbf{w}_1$ and $\frac{\partial \mathbf{w}}{\partial \mathbf{v}_2} = \mathbf{w}_2$ as $\frac{\partial \mathbf{w}}{\partial \{\mathbf{v}_1, \mathbf{v}_2\}} = \{\mathbf{w}_1, \mathbf{w}_2\}$. $\frac{\partial \mathbf{w}}{\partial \mathbf{v}} \mathrel{+}= \mathbf{A}$ adds the 3×3 block $\mathbf{A}$ to the rows corresponding to $\mathbf{w}$ and columns corresponding to the global indices of vertex $\mathbf{v}$. $\mathbf{O}_{10\times3n}$ denotes the zero matrix with 10 rows and $3n$ columns.

References

EBERLY, D. H. 2003. *Game Physics*. Elsevier Science Inc.

Algorithm 1 Mass properties of a triangulated solid

$\mathbf{s}_\Omega = \mathbf{o}_{10}$
for all $i \in \mathcal{T}$ **do**
  $\mathbf{u} = (\mathbf{b} - \mathbf{a})$
  $\mathbf{v} = (\mathbf{c} - \mathbf{a})$
  $\mathbf{n} = \mathbf{u} \times \mathbf{v}$
  $\mathbf{h}_1 = \mathbf{a} + \mathbf{b} + \mathbf{c}$
  $\mathbf{h}_2 = \mathbf{a} * \mathbf{a} + \mathbf{b} * (\mathbf{a} + \mathbf{b})$
  $\mathbf{h}_3 = \mathbf{h}_2 + \mathbf{c} * \mathbf{h}_1$
  $\mathbf{h}_4 = \mathbf{a} * \mathbf{a} * \mathbf{a} + \mathbf{b} * \mathbf{h}_2 + \mathbf{c} * \mathbf{h}_3$
  $\mathbf{h}_5 = \mathbf{h}_3 + \mathbf{a} * (\mathbf{h}_1 + \mathbf{a})$
  $\mathbf{h}_6 = \mathbf{h}_3 + \mathbf{b} * (\mathbf{h}_1 + \mathbf{b})$
  $\mathbf{h}_7 = \mathbf{h}_3 + \mathbf{c} * (\mathbf{h}_1 + \mathbf{c})$
  $\mathbf{h}_8 = \bar{\mathbf{a}} * \mathbf{h}_5 + \bar{\mathbf{b}} * \mathbf{h}_6 + \bar{\mathbf{c}} * \mathbf{h}_7$
  $s_1 \mathrel{+}= \mathbf{e}_x \cdot (\mathbf{n} * \mathbf{h}_1)$
  $[s_x, s_y, s_z] \mathrel{+}= \mathbf{n} * \mathbf{h}_3$
  $[s_{xy}, s_{yz}, s_{xz}] \mathrel{+}= \mathbf{n} * \mathbf{h}_8$
  $[s_{x^2}, s_{y^2}, s_{z^2}] \mathrel{+}= \mathbf{n} * \mathbf{h}_4$
end for
$s_1 \mathrel{*}= \frac{1}{6}$
$[s_x, s_y, s_z] \mathrel{*}= \frac{1}{24}$
$[s_{xy}, s_{yz}, s_{xz}] \mathrel{*}= \frac{1}{120}$
$[s_{x^2}, s_{y^2}, s_{z^2}] \mathrel{*}= \frac{1}{60}$
$\mathbf{s}_\Omega \mathrel{*}= \rho$

Algorithm 2 Mass property derivatives of a triangulated solid

$\frac{\partial \mathbf{s}_\Omega}{\partial \mathbf{V}} = \mathbf{O}_{10 \times 3n}$
for all $i \in \mathcal{T}$ **do**
  $\frac{\partial \mathbf{n}}{\partial \mathbf{a}} = [\mathbf{v}]_\times + [\mathbf{u}]_\times^T, \;\; \frac{\partial \mathbf{n}}{\partial \mathbf{b}} = [\mathbf{v}]_\times^T, \;\; \frac{\partial \mathbf{n}}{\partial \mathbf{c}} = [\mathbf{u}]_\times$
  $\mathbf{d}_{3,\mathbf{a}} = 2\mathbf{a}+\mathbf{b}+\mathbf{c}, \quad \mathbf{d}_{3,\mathbf{b}} = \mathbf{a}+2\mathbf{b}+\mathbf{c}, \quad \mathbf{d}_{3,\mathbf{c}} = \mathbf{a}+\mathbf{b}+2\mathbf{c}$
  $\mathbf{d}_{5,\mathbf{a}} = 6\mathbf{a} + 2\mathbf{b} + 2\mathbf{c}$
  $\mathbf{d}_{6,\mathbf{b}} = 2\mathbf{a} + 6\mathbf{b} + 2\mathbf{c}$
  $\mathbf{d}_{7,\mathbf{c}} = 2\mathbf{a} + 2\mathbf{b} + 6\mathbf{c}$
  $\mathbf{d}_{5,\mathbf{b}} = \mathbf{d}_{6,\mathbf{a}} = 2\mathbf{a} + 2\mathbf{b} + \mathbf{c}$
  $\mathbf{d}_{5,\mathbf{c}} = \mathbf{d}_{7,\mathbf{a}} = 2\mathbf{a} + \mathbf{b} + 2\mathbf{c}$
  $\mathbf{d}_{6,\mathbf{c}} = \mathbf{d}_{7,\mathbf{b}} = \mathbf{a} + 2\mathbf{b} + 2\mathbf{c}$
  $\mathbf{d}_{8,\mathbf{a}} = \bar{\mathbf{a}} * \mathbf{d}_{5,\mathbf{a}} + \bar{\mathbf{b}} * \mathbf{d}_{5,\mathbf{b}} + \bar{\mathbf{c}} * \mathbf{d}_{5,\mathbf{c}}$
  $\mathbf{d}_{8,\mathbf{b}} = \bar{\mathbf{a}} * \mathbf{d}_{6,\mathbf{a}} + \bar{\mathbf{b}} * \mathbf{d}_{6,\mathbf{b}} + \bar{\mathbf{c}} * \mathbf{d}_{6,\mathbf{c}}$
  $\mathbf{d}_{8,\mathbf{c}} = \bar{\mathbf{a}} * \mathbf{d}_{7,\mathbf{a}} + \bar{\mathbf{b}} * \mathbf{d}_{7,\mathbf{b}} + \bar{\mathbf{c}} * \mathbf{d}_{7,\mathbf{c}}$
  $\frac{\partial s_1}{\partial \{\mathbf{a},\mathbf{b},\mathbf{c}\}} \mathrel{+}= \mathbf{e}_x^T \left(\frac{\partial \mathbf{n}}{\partial \{\mathbf{a},\mathbf{b},\mathbf{c}\}} \mathbf{H}_1^T + \mathrm{diag}(\mathbf{n}) \right)$
  $\frac{\partial [s_x, s_y, s_z]}{\partial \{\mathbf{a},\mathbf{b},\mathbf{c}\}} \mathrel{+}= \frac{\partial \mathbf{n}}{\partial \{\mathbf{a},\mathbf{b},\mathbf{c}\}} \mathbf{H}_3^T + \mathrm{diag}(\mathbf{n} * \{\mathbf{d}_{3,\mathbf{a}}, \mathbf{d}_{3,\mathbf{b}}, \mathbf{d}_{3,\mathbf{c}}\})$
  $\frac{\partial [s_{xy}, s_{yz}, s_{xz}]}{\partial \mathbf{a}} \mathrel{+}= \frac{\partial \mathbf{n}}{\partial \mathbf{a}} \mathbf{H}_8^T + \mathrm{diag}(\mathbf{n} * \mathbf{d}_{8,\mathbf{a}}) + [\mathbf{n}\mathbf{h}_5]_*$
  $\frac{\partial [s_{xy}, s_{yz}, s_{xz}]}{\partial \mathbf{b}} \mathrel{+}= \frac{\partial \mathbf{n}}{\partial \mathbf{b}} \mathbf{H}_8^T + \mathrm{diag}(\mathbf{n} * \mathbf{d}_{8,\mathbf{b}}) + [\mathbf{n}\mathbf{h}_6]_*$
  $\frac{\partial [s_{xy}, s_{yz}, s_{xz}]}{\partial \mathbf{c}} \mathrel{+}= \frac{\partial \mathbf{n}}{\partial \mathbf{c}} \mathbf{H}_8^T + \mathrm{diag}(\mathbf{n} * \mathbf{d}_{8,\mathbf{c}}) + [\mathbf{n}\mathbf{h}_7]_*$
  $\frac{\partial [s_{x^2}, s_{y^2}, s_{z^2}]}{\partial \{\mathbf{a},\mathbf{b},\mathbf{c}\}} \mathrel{+}= \frac{\partial \mathbf{n}}{\partial \{\mathbf{a},\mathbf{b},\mathbf{c}\}} \mathbf{H}_4^T + \mathrm{diag}(\mathbf{n} * \{\mathbf{h}_5, \mathbf{h}_6, \mathbf{h}_7\})$
end for
$\frac{\partial s_1}{\partial \mathbf{V}} \mathrel{*}= \frac{1}{6}$
$\frac{\partial [s_x, s_y, s_z]}{\partial \mathbf{V}} \mathrel{*}= \frac{1}{24}$
$\frac{\partial [s_{xy}, s_{yz}, s_{xz}]}{\partial \mathbf{V}} \mathrel{*}= \frac{1}{120}$
$\frac{\partial [s_{x^2}, s_{y^2}, s_{z^2}]}{\partial \mathbf{V}} \mathrel{*}= \frac{1}{60}$
$\frac{\partial \mathbf{s}_\Omega}{\partial \mathbf{V}} \mathrel{*}= \rho$