Vibration-Minimizing Motion Retargeting for Robotic Characters

SHAYAN HOSHYARI, Disney Research and University of British Columbia
HONGYI XU, Disney Research
ESPEN KNOOP, Disney Research
STELIAN COROS, ETH Zurich
MORITZ BÄCHER, Disney Research

In our supplemental material, we derive the relative coordinate formulation of our equations of motion (Sec. 1), and derive the adjoint system that we need to compute analytical gradients for our retargeting optimization (Sec. 2). Additional validations and results are discussed in Sec. 3.

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1 DERIVATION OF RELATIVE COORDINATE FORMULATION

The deformed configuration is

$$\mathbf{x}(\mathbf{X}, t) = \mathbf{R}(t)\,[\mathbf{X} + \mathbf{\Phi}(\mathbf{X})\mathbf{u}(t)] + \mathbf{c}(t). \tag{1}$$

To derive the corresponding velocity and acceleration, we drop the arguments $\mathbf{x} = \mathbf{R}\,(\mathbf{X} + \mathbf{\Phi u}) + \mathbf{c}$.

The velocity of the deformed configuration is

$$\begin{aligned}\dot{\mathbf{x}} &= \dot{\mathbf{R}}\,(\mathbf{X} + \mathbf{\Phi u}) + \mathbf{R\Phi}\dot{\mathbf{u}} + \dot{\mathbf{c}} \\ &= [\boldsymbol{\omega}]_\times\,\mathbf{R}\,(\mathbf{X} + \mathbf{\Phi u}) + \mathbf{R\Phi v} + \mathbf{w},\end{aligned} \tag{2}$$

and its acceleration

$$\begin{aligned}\ddot{\mathbf{x}} &= \ddot{\mathbf{R}}\,(\mathbf{X} + \mathbf{\Phi u}) + 2\dot{\mathbf{R}}\mathbf{\Phi}\dot{\mathbf{u}} + \mathbf{R\Phi}\ddot{\mathbf{u}} + \ddot{\mathbf{c}} \\ &= \left([\dot{\boldsymbol{\omega}}]_\times\,\mathbf{R} + [\boldsymbol{\omega}]^2_\times\,\mathbf{R}\right)(\mathbf{X} + \mathbf{\Phi u}) + 2\,[\boldsymbol{\omega}]_\times\,\mathbf{R\Phi v} + \mathbf{R\Phi}\dot{\mathbf{v}} + \dot{\mathbf{w}} \\ &= \mathbf{R\Phi}\dot{\mathbf{v}} + [\dot{\boldsymbol{\omega}}]_\times\,\mathbf{R}\,(\mathbf{X} + \mathbf{\Phi u}) + \dot{\mathbf{w}} + [\boldsymbol{\omega}]^2_\times\,\mathbf{R}\,(\mathbf{X} + \mathbf{\Phi u}) + 2\,[\boldsymbol{\omega}]_\times\,\mathbf{R\Phi v} \\ &= \underbrace{\mathbf{R\Phi}\dot{\mathbf{v}}}_{db\ acc.}\ \underbrace{-\mathbf{R}\,([\mathbf{X}]_\times + [\mathbf{\Phi u}]_\times)\,\mathbf{R}^T\dot{\boldsymbol{\omega}} + \dot{\mathbf{w}}}_{rb\ acc.} \\ &\quad + \underbrace{\mathbf{R}[\mathbf{R}^T\boldsymbol{\omega}]^2_\times\,(\mathbf{X} + \mathbf{\Phi u})}_{centrifugal\ acc.} + \underbrace{2\mathbf{R}[\mathbf{R}^T\boldsymbol{\omega}]_\times\mathbf{\Phi v}}_{Coriolis\ acc.}.\end{aligned} \tag{3}$$

In above derivations, we make use of identities $\dot{\mathbf{R}} = [\boldsymbol{\omega}]_\times\,\mathbf{R}$ and $\ddot{\mathbf{R}} = [\dot{\boldsymbol{\omega}}]_\times\,\mathbf{R} + [\boldsymbol{\omega}]^2_\times\,\mathbf{R}$ in the first three lines. Because $[\mathbf{R}^T\mathbf{a}]_\times\mathbf{b} = \mathbf{R}^T\mathbf{a} \times \mathbf{R}^T\mathbf{R}\mathbf{b} = \mathbf{R}^T[\mathbf{a}]_\times\mathbf{R}\mathbf{b}$, the identity $\mathbf{R}^T[\mathbf{a}]_\times\mathbf{R} = [\mathbf{R}^T\mathbf{a}]_\times$ holds.

Authors' addresses: Shayan Hoshyari, Disney Research and University of British Columbia; Hongyi Xu, Disney Research; Espen Knoop, Disney Research; Stelian Coros, ETH Zurich; Moritz Bächer, Disney Research.

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For acceleration terms that have subexpressions $[\mathbf{a}]_\times\mathbf{R}\mathbf{b}$, we make use of the latter identity to move the rotation matrix to the left $\mathbf{R}\mathbf{R}^T[\mathbf{a}]_\times\mathbf{R}\mathbf{b} = \mathbf{R}[\mathbf{R}^T\mathbf{a}]_\times\mathbf{b}$. For the rigid body acceleration term, we further make use of identities $[\mathbf{a}]_\times\mathbf{b} = -[\mathbf{b}]_\times\mathbf{a}$ and $[\mathbf{a} + \mathbf{b}]_\times = [\mathbf{a}]_\times + [\mathbf{b}]_\times$.

To form the inertial forces,

$$\int_\Omega \mathbf{\Phi}^T\mathbf{R}^T\rho\,\ddot{\mathbf{x}}\,\mathrm{d}\mathbf{X} = \int_\Omega \rho\,\mathbf{\Phi}^T\mathbf{R}^T\ddot{\mathbf{x}}\,\mathrm{d}\mathbf{X},$$

we integrate the individual acceleration terms. When integrating the acceleration of the deformable body (*db acc.*), we see why it is useful to multiply with the transpose of the rigid body rotation

$$\int_\Omega \rho\,\mathbf{\Phi}^T\mathbf{R}^T\mathbf{R\Phi}\dot{\mathbf{v}}\,\mathrm{d}\mathbf{X} = \left(\int_\Omega \rho\,\mathbf{\Phi}^T\mathbf{\Phi}\,\mathrm{d}\mathbf{X}\right)\dot{\mathbf{v}} = \underline{\mathbf{M}}\dot{\mathbf{v}}. \tag{4}$$

Integration results in the same mass matrix as for the absolute coordinate formulation ($\underline{\mathbf{M}} \in \mathbb{R}^{3n\times 3n}$).

Integration of the rigid body acceleration (*rb acc.*) leads to a force

$$-\left(\underline{\mathbf{M1}} + \underline{\mathbf{M2}}(\mathbf{u})\right)\mathbf{R}^T\dot{\boldsymbol{\omega}} + \underline{\mathbf{M3}}\,\mathbf{R}^T\dot{\mathbf{w}} \tag{5}$$

that depends on a total of three mass matrices

$$\underline{\mathbf{M1}} = \int_\Omega \rho\,\mathbf{\Phi}^T[\mathbf{X}]_\times\,\mathrm{d}\mathbf{X} \qquad \underline{\mathbf{M2}}(\mathbf{u}) = \int_\Omega \rho\,\mathbf{\Phi}^T[\mathbf{\Phi u}]_\times\,\mathrm{d}\mathbf{X}$$
$$\underline{\mathbf{M3}} = \int_\Omega \rho\,\mathbf{\Phi}^T\,\mathrm{d}\mathbf{X}$$

where $\underline{\mathbf{M1}} \in \mathbb{R}^{3n\times 3}$ and $\underline{\mathbf{M3}} \in \mathbb{R}^{3n\times 3}$ are constant and $\underline{\mathbf{M2}} \in \mathbb{R}^{3n\times 3}$ depends on the displacement. To efficiently compute $\underline{\mathbf{M2}}$, we substitute $\sum_{k=1}^{3n} \boldsymbol{\phi}_k u_k$ for the displacement $\mathbf{\Phi u}$

$$\underline{\mathbf{M2}}(\mathbf{u}) = \sum_{k=1}^{3n}\left(\int_\Omega \rho\,\mathbf{\Phi}^T[\boldsymbol{\phi}_k]_\times\,\mathrm{d}\mathbf{X}\right)u_k \tag{6}$$

and precompute the $3n \times 3$ blocks in brackets.

To derive the centrifugal forces, we first split the corresponding acceleration (*centrifugal acc.*) into two terms

$$\mathbf{R}\left((\mathbf{R}^T\boldsymbol{\omega})\cdot(\mathbf{X} + \mathbf{\Phi u})\right)\mathbf{R}^T\boldsymbol{\omega} - \mathbf{R}(\mathbf{X} + \mathbf{\Phi u})\,(\boldsymbol{\omega}\cdot\boldsymbol{\omega}) \tag{7}$$

where we apply the identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$ to the subexpression $[\mathbf{R}^T\boldsymbol{\omega}]^2_\times(\mathbf{X} + \mathbf{\Phi u}) = (\mathbf{R}^T\boldsymbol{\omega}) \times (\mathbf{R}^T\boldsymbol{\omega}) \times (\mathbf{X} + \mathbf{\Phi u})$.

To integrate the first term, we split the integral into a sum of integrals and look at the transpose of the j-th column of $\mathbf{\Phi}$

$$\sum_{k=1}^{3n} \int_\Omega \rho\, \boldsymbol{\phi}_j^T \left((\mathbf{R}^T \boldsymbol{\omega}) \cdot (\mathbf{X} + \boldsymbol{\phi}_k u_k) \right) \mathbf{R}^T \boldsymbol{\omega}\, \mathrm{d}\mathbf{X} \tag{8}$$

$$= \boldsymbol{\omega}^T \mathbf{R} \left(\sum_{k=1}^{3n} \int_\Omega \rho\, (\mathbf{X} + \boldsymbol{\phi}_k u_k) \boldsymbol{\phi}_j^T\, \mathrm{d}\mathbf{X} \right) \mathbf{R}^T \boldsymbol{\omega}$$

$$= \boldsymbol{\omega}^T \mathbf{R} \left(\left(\int_\Omega \rho\, \mathbf{X} \boldsymbol{\phi}_j^T\, \mathrm{d}\mathbf{X} \right) + \sum_{k=1}^{3n} \left(\int_\Omega \rho\, \boldsymbol{\phi}_k \boldsymbol{\phi}_j^T\, \mathrm{d}\mathbf{X} \right) u_k \right) \mathbf{R}^T \boldsymbol{\omega}.$$

The derivation of the second term is straightforward.

To form the resulting centrifugal forces

$$\mathbf{f}_{\mathrm{cen}}(\mathbf{q}, \mathbf{u}, \boldsymbol{\omega}) = \sum_{j=1}^{3n} \boldsymbol{\omega}^T \mathbf{R} \left(\underline{\mathbf{M4}}_j + \underline{\mathbf{M5}}_j(\mathbf{u}) \right) \mathbf{R}^T \boldsymbol{\omega}\, \mathbf{e}_j \tag{9}$$
$$- \left(\underline{\mathbf{M6}} + \underline{\mathbf{M}}\mathbf{u} \right) (\boldsymbol{\omega} \cdot \boldsymbol{\omega})$$

where $\mathbf{e}_j$ is the j-th column of the $3n \times 3n$ identity matrix, we precompute the additional mass matrices

$$\underline{\mathbf{M4}}_j = \int_\Omega \rho\, \mathbf{X} \boldsymbol{\phi}_j^T\, \mathrm{d}\mathbf{X} \qquad \underline{\mathbf{M5}}_j(\mathbf{u}) = \sum_{k=1}^{3n} \left(\int_\Omega \rho\, \boldsymbol{\phi}_k \boldsymbol{\phi}_j^T\, \mathrm{d}\mathbf{X} \right) u_k$$

$$\underline{\mathbf{M6}} = \int_\Omega \rho\, \mathbf{\Phi}^T \mathbf{X}\, \mathrm{d}\mathbf{X}.$$

Analogously to $\underline{\mathbf{M2}}$, we precompute the blocks in brackets for matrix $\underline{\mathbf{M5}}_j$. Because $\mathbf{\Phi}$ has $3n$ columns, there are $3n$ 3×3 matrices $\underline{\mathbf{M4}}_j$ and $\underline{\mathbf{M5}}_j(\mathbf{u})$ ($j = 1, \ldots, 3n$). $\underline{\mathbf{M6}}$ is a $3n$ *vector.*

To derive the Coriolis force

$$\mathbf{f}_{\mathrm{cor}}(\mathbf{q}, \boldsymbol{\omega}, \mathbf{v}) = -2\underline{\mathbf{M2}}(\mathbf{v})\, \mathbf{R}^T \boldsymbol{\omega}, \tag{10}$$

we substitute $[\mathbf{R}^T \boldsymbol{\omega}]_\times$ for $\mathbf{a}$ and $\mathbf{\Phi v}$ for $\mathbf{b}$ in the identity $[\mathbf{a}]_\times \mathbf{b} = -[\mathbf{b}]_\times \mathbf{a}$ in the Coriolis term (*Coriolis acc.*) prior to integration. For this term, we reuse $\underline{\mathbf{M2}}$, setting its parameter to the velocities $\mathbf{v}$ instead of the displacements $\mathbf{u}$.

Note that the fictitious centrifugal and Coriolis forces depend on $\mathbf{q}$ because we use quaternions instead of rotations. In our implementation, we extract rotations from quaternions $\mathbf{R}(\mathbf{q})$.

1.1 Reduced Basis

In a reduced formulation, the displacements are defined as

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{\Phi}(\mathbf{X}) \mathbf{U}_\mathrm{r} \mathbf{u}_\mathrm{r}(t). \tag{11}$$

To derive the reduced mass matrices and inertial forces, we can replace the full basis $\mathbf{\Phi}(\mathbf{X})$ with the reduced basis $\mathbf{\Phi}_\mathrm{r}(\mathbf{X}) = \mathbf{\Phi}(\mathbf{X})\mathbf{U}_\mathrm{r}$ in above derivations. We again drop arguments. $\mathbf{\Phi}_\mathrm{r}$ is now a $3 \times r$-matrix, $\mathbf{u}_\mathrm{r}$ a r-vector, and $\boldsymbol{\phi}_{r,k} \in \mathbb{R}^3$ the k-th column of $\mathbf{\Phi}_\mathrm{r}$.

The mass matrices are

$$\underline{\mathbf{M}}_\mathrm{r} = \mathbf{U}_\mathrm{r}^T \underline{\mathbf{M}} \mathbf{U}_\mathrm{r} = \mathbf{E}_{r \times r} \tag{12}$$

$$\underline{\mathbf{M1}}_\mathrm{r} = \mathbf{U}_\mathrm{r}^T \underline{\mathbf{M1}} \tag{13}$$

$$\underline{\mathbf{M2}}_\mathrm{r}(\mathbf{u}_\mathrm{r}) = \left(\sum_{k=1}^{r} \left(\int_\Omega \rho\, \mathbf{\Phi}_\mathrm{r}^T [\boldsymbol{\phi}_{\mathrm{r},k}]_\times\, \mathrm{d}\mathbf{X} \right) u_{\mathrm{r},k} \right) \tag{14}$$

$$\underline{\mathbf{M3}}_\mathrm{r} = \mathbf{U}_\mathrm{r}^T \underline{\mathbf{M3}} \tag{15}$$

$$\underline{\mathbf{M4}}_{\mathrm{r},j} = \int_\Omega \rho\, \mathbf{X} \boldsymbol{\phi}_{\mathrm{r},j}^T\, \mathrm{d}\mathbf{X} \tag{16}$$

$$\underline{\mathbf{M5}}_{\mathrm{r},j}(\mathbf{u}_\mathrm{r}) = \sum_{k=1}^{r} \left(\int_\Omega \rho\, \boldsymbol{\phi}_{\mathrm{r},k} \boldsymbol{\phi}_{\mathrm{r},j}^T\, \mathrm{d}\mathbf{X} \right) u_{\mathrm{r},k} \tag{17}$$

$$\underline{\mathbf{M6}}_\mathrm{r} = \mathbf{U}_\mathrm{r}^T \underline{\mathbf{M6}} \tag{18}$$

where we use numerical integration to precompute the blocks in brackets for $\underline{\mathbf{M2}}_\mathrm{r}(\mathbf{u}_\mathrm{r})$ and $\underline{\mathbf{M5}}_{\mathrm{r},j}(\mathbf{u}_\mathrm{r})$, and the j-th 3×3 matrices $\underline{\mathbf{M4}}_{\mathrm{r},j}$.

The inertial forces corresponding to the deformable body (*db acc.*) reduce to

$$\mathbf{U}_\mathrm{r}^T \underline{\mathbf{M}} \mathbf{U}_\mathrm{r} \dot{\mathbf{v}}_\mathrm{r} = \dot{\mathbf{v}}_\mathrm{r}, \tag{19}$$

and for the rigid body (*rb acc.*) to

$$- \left(\underline{\mathbf{M1}}_\mathrm{r} + \underline{\mathbf{M2}}_\mathrm{r}(\mathbf{u}_\mathrm{r}) \right) \mathbf{R}^T \dot{\boldsymbol{\omega}} + \underline{\mathbf{M3}}_\mathrm{r}\, \mathbf{R}^T \dot{\mathbf{w}}. \tag{20}$$

The centrifugal force becomes

$$\mathbf{f}_{\mathrm{cen}}(\mathbf{q}, \mathbf{u}_\mathrm{r}, \boldsymbol{\omega}) = \sum_{j=1}^{r} \boldsymbol{\omega}^T \mathbf{R} \left(\underline{\mathbf{M4}}_{\mathrm{r},j} + \underline{\mathbf{M5}}_{\mathrm{r},j}(\mathbf{u}) \right) \mathbf{R}^T \boldsymbol{\omega}\, \mathbf{e}_j \tag{21}$$
$$- \left(\underline{\mathbf{M6}} + \underline{\mathbf{M}}\mathbf{u} \right) (\boldsymbol{\omega} \cdot \boldsymbol{\omega})$$

where $\mathbf{e}_j$ is the j-th column of the $r \times r$ identity matrix, and the reduced Coriolis force is

$$\mathbf{f}_{\mathrm{cor}}(\mathbf{q}, \boldsymbol{\omega}, \mathbf{v}_\mathrm{r}) = -2\underline{\mathbf{M2}}_\mathrm{r}(\mathbf{v}_\mathrm{r})\, \mathbf{R}^T \boldsymbol{\omega}. \tag{22}$$

2 DERIVATION OF THE ADJOINT SYSTEM

To solve our retargeting problem

$$\min_{\mathbf{p}} G(\mathbf{p}, \mathbf{U}(\mathbf{p})) + R(\mathbf{p}) \tag{23}$$

$$\text{subject to} \quad \mathbf{G}(t, \mathbf{p}, \mathbf{S}(t, \mathbf{p}), \dot{\mathbf{S}}(t, \mathbf{p})) = 0 \quad \text{and} \quad \mathbf{S}(0) = \mathbf{S}_0(\mathbf{p}),$$

we need to be able to compute an analytical gradient $\frac{\mathrm{d}G(\mathbf{p}, \mathbf{U}(\mathbf{p}))}{\mathrm{d}\mathbf{p}}$.

We largely follow the derivation in the work of Cao et al. [2003], omitting terms that are not relevant in our context. While our semi-implicit DAE system can easily be brought into standard Hessenberg index-2 form, we prefer to keep the mass matrix $\mathbf{M}$ on the left-hand side because it simplifies the adjoint DAE.

To keep the derivation general, we assume our objective to depend on the state $\mathbf{S}$ for the first part of the derivation

$$G(\mathbf{p}, \mathbf{S}(\mathbf{p})) = \int_0^T g(t, \mathbf{p}, \mathbf{S}(\mathbf{p}))\, \mathrm{d}t. \tag{24}$$

The augmented objective with *continuous, time-dependent* Lagrange multipliers $\boldsymbol{\lambda}(t)$ for above problem is

$$I(\mathbf{p}, \mathbf{S}) = G(\mathbf{p}, \mathbf{S}) - \int_0^T \boldsymbol{\lambda}^T \mathbf{G}(t, \mathbf{p}, \mathbf{S}, \dot{\mathbf{S}})\, \mathrm{d}t. \tag{25}$$

Because our DAE system is satisfied at every $t \in [0, T]$, the total derivative of G and I are equivalent, hence

$$\frac{\mathrm{d}G}{\mathrm{d}\mathbf{p}} = \int_0^T (g_\mathbf{p} + g_\mathbf{S} \mathbf{S}_\mathbf{p})\, \mathrm{d}t - \int_0^T \boldsymbol{\lambda}^T (\mathbf{G}_\mathbf{p} + \mathbf{G}_\mathbf{S} \mathbf{S}_\mathbf{p} + \mathbf{G}_{\dot{\mathbf{S}}} \dot{\mathbf{S}}_\mathbf{p})\, \mathrm{d}t \tag{26}$$

where we use subscripts for partial derivatives. Note that $\mathbf{S}_\mathbf{p}$ and $\dot{\mathbf{S}}_\mathbf{p}$ are total derivatives.

Integration by parts of the term $\boldsymbol{\lambda}^T \mathbf{G}_{\dot{\mathbf{S}}} \dot{\mathbf{S}}_\mathbf{p}$

$$\int_0^T \boldsymbol{\lambda}^T \mathbf{G}_{\dot{\mathbf{S}}} \dot{\mathbf{S}}_\mathbf{p}\, \mathrm{d}t = \left(\boldsymbol{\lambda}^T \mathbf{G}_{\dot{\mathbf{S}}} \mathbf{S}_\mathbf{p} \right)\Big|_0^T - \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \left(\boldsymbol{\lambda}^T \mathbf{G}_{\dot{\mathbf{S}}} \right) \mathbf{S}_\mathbf{p}\, \mathrm{d}t \tag{27}$$

enables us to turn the dependence on $\dot{\mathbf{S}}_\mathbf{p}$ into a dependence on $\mathbf{S}_\mathbf{p}$

$$\frac{\mathrm{d}G}{\mathrm{d}\mathbf{p}} = \int_0^T (g_\mathbf{p} - \boldsymbol{\lambda}^T \mathbf{G}_\mathbf{p})\,\mathrm{d}t - \int_0^T \left(-g_\mathbf{S} + \boldsymbol{\lambda}^T \mathbf{G}_\mathbf{S} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{\lambda}^T \mathbf{G}_{\dot{\mathbf{S}}}\right)\right) \mathbf{S}_\mathbf{p}\,\mathrm{d}t - \left(\boldsymbol{\lambda}^T \mathbf{G}_{\dot{\mathbf{S}}} \mathbf{S}_\mathbf{p}\right)\Big|_0^T. \tag{28}$$

By setting the term in brackets of the second integrand to zero, we then form the adjoint system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{\lambda}^T \mathbf{G}_{\dot{\mathbf{S}}}\right) = \boldsymbol{\lambda}^T \mathbf{G}_\mathbf{S} + g_\mathbf{S}. \tag{29}$$

Applying the chain rule and transposing the system, we form the adjoint DAE

$$\mathbf{G}_{\dot{\mathbf{S}}}^T \dot{\boldsymbol{\lambda}} = \left(-\left(\frac{\mathrm{d}\mathbf{G}_{\dot{\mathbf{S}}}}{\mathrm{d}t}\right)^T + \mathbf{G}_\mathbf{S}^T\right)\boldsymbol{\lambda} + g_\mathbf{S}^T. \tag{30}$$

For our particular DAE system

$$\mathbf{G} = \begin{bmatrix} \dot{\mathbf{U}} - \mathbf{T}(\mathbf{U})\mathbf{V} \\ \mathbf{M}(\mathbf{U})\dot{\mathbf{V}} - \mathbf{F}(\mathbf{U},\mathbf{V}) - (C_\mathbf{U}(t,\mathbf{U})\mathbf{T}(\mathbf{U}))^T \boldsymbol{\Lambda} \\ C_t(t,\mathbf{U}) + C_\mathbf{U}(t,\mathbf{U})\mathbf{T}(\mathbf{U})\mathbf{V} + \alpha C(t,\mathbf{U}) \end{bmatrix} = 0, \tag{31}$$

the Jacobian w.r.t. the state is

$$\mathbf{G}_\mathbf{S} = \begin{bmatrix} \mathbf{G}_\mathbf{U}(t,\mathbf{U},\mathbf{V}) & \mathbf{G}_\mathbf{V}(t,\mathbf{U},\mathbf{V}) & \mathbf{G}_\boldsymbol{\Lambda}(t,\mathbf{U}) \end{bmatrix} \tag{32}$$

with columns

$$\mathbf{G}_\mathbf{U} = \begin{bmatrix} -\mathbf{T}_\mathbf{U}(\mathbf{U})\mathbf{V} \\ \mathbf{M}_\mathbf{U}(\mathbf{U})\dot{\mathbf{V}} - \mathbf{F}_\mathbf{U}(\mathbf{U},\mathbf{V}) - \left(C_{\mathbf{U},\mathbf{U}}(t,\mathbf{U})\mathbf{T}(\mathbf{U}) + C_\mathbf{U}(t,\mathbf{U})\mathbf{T}_\mathbf{U}(\mathbf{U})\right)^T \\ C_{t,\mathbf{U}}(t,\mathbf{U}) + \left(C_{\mathbf{U},\mathbf{U}}(t,\mathbf{U})\mathbf{T}(\mathbf{U}) + C_\mathbf{U}(t,\mathbf{U})\mathbf{T}_\mathbf{U}(\mathbf{U})\right)\mathbf{V} + \alpha C(t,\mathbf{U}) \end{bmatrix},$$

$$\mathbf{G}_\mathbf{V} = \begin{bmatrix} -\mathbf{T}(\mathbf{U}) \\ -\mathbf{F}_\mathbf{V}(\mathbf{U},\mathbf{V}) \\ C_\mathbf{U}(t,\mathbf{U})\mathbf{T}(\mathbf{U}) \end{bmatrix}, \quad \text{and} \quad \mathbf{G}_\boldsymbol{\Lambda} = \begin{bmatrix} \\ -\left(C_\mathbf{U}(t,\mathbf{U})\mathbf{T}(\mathbf{U})\right)^T \\ \\ \end{bmatrix}$$

where $C_{t,\mathbf{U}} = \frac{\partial^2 C}{\partial t \partial \mathbf{U}}$ and $C_{\mathbf{U},\mathbf{U}} = \frac{\partial^2 C}{\partial \mathbf{U}^2}$.

The Jacobian w.r.t. the time-derivative of the state is

$$\mathbf{G}_{\dot{\mathbf{S}}} = \begin{bmatrix} \mathbf{E} & & \\ & \mathbf{M}(\mathbf{U}) & \\ & & \end{bmatrix}. \tag{33}$$

And its time-derivative is

$$\frac{\mathrm{d}\mathbf{G}_{\dot{\mathbf{S}}}}{\mathrm{d}t} = \begin{bmatrix} & & \\ & \mathbf{M}_\mathbf{U}(\mathbf{U})\dot{\mathbf{U}} & \\ & & \end{bmatrix}, \tag{34}$$

where only the "center" element of the 3-by-3 block matrix is non-zero.

For our particular system, g only depends on generalized positions and velocities

$$g_\mathbf{S} = \begin{bmatrix} g_\mathbf{U} & g_\mathbf{V} & \end{bmatrix}. \tag{35}$$

In summary, our *linear adjoint DAE* is

$$\begin{bmatrix} \mathbf{E} & & \\ & \mathbf{M}^T & \\ & & \end{bmatrix} \dot{\boldsymbol{\lambda}} = \left(-\begin{bmatrix} & & \\ & \dot{\mathbf{U}}^T \mathbf{M}_\mathbf{U}^T & \\ & & \end{bmatrix} + \begin{bmatrix} \mathbf{G}_\mathbf{U}^T \\ \mathbf{G}_\mathbf{V}^T \\ \mathbf{G}_\boldsymbol{\Lambda}^T \end{bmatrix}\right)\boldsymbol{\lambda} + \begin{bmatrix} g_\mathbf{U}^T \\ g_\mathbf{V}^T \\ \end{bmatrix}.$$

It remains to discuss initial conditions. To evaluate the gradient $\frac{\mathrm{d}G}{\mathrm{d}\mathbf{p}}$, we need to know

$$\left(\boldsymbol{\lambda}^T \mathbf{G}_{\dot{\mathbf{S}}} \mathbf{S}_\mathbf{p}\right)\Big|_{t=0} \quad \text{and} \quad \left(\boldsymbol{\lambda}^T \mathbf{G}_{\dot{\mathbf{S}}} \mathbf{S}_\mathbf{p}\right)\Big|_{t=T}. \tag{36}$$

At time $t = 0$, the Jacobian $\mathbf{S}_\mathbf{p}$ is equal to the analytical derivative $\frac{\mathrm{d}\mathbf{S}_0}{\mathrm{d}\mathbf{p}}$ of our initial conditions $\mathbf{S}_0$. If we set the *initial conditions* for our adjoint DAE to be $\boldsymbol{\lambda}(T) = 0$, then

$$\left(\boldsymbol{\lambda}^T \mathbf{G}_{\dot{\mathbf{S}}} \mathbf{S}_\mathbf{p}\right)\Big|_{t=T} = 0. \tag{37}$$

Note that the initial conditions $\mathbf{S}_0$ for our DAE are *not* dependent on $\mathbf{p}$ because we assume the system to be at rest at the start of an animation. Hence, $\mathbf{S}_\mathbf{p}$ is zero at time $t = 0$, and therefore

$$\left(\boldsymbol{\lambda}^T \mathbf{G}_{\dot{\mathbf{S}}} \mathbf{S}_\mathbf{p}\right)\Big|_{t=0} = 0. \tag{38}$$

If objective g depends on the algebraic variables $\boldsymbol{\Lambda}$, the initial conditions for our adjoint DAE, $\boldsymbol{\lambda}(T) = 0$, are in conflict with the adjoint DAE, and an additional treatment is necessary [Cao et al. 2003].

In summary, the *analytical gradient* of our objective is

$$\frac{\mathrm{d}G}{\mathrm{d}\mathbf{p}} = \int_0^T (g_\mathbf{p} - \boldsymbol{\lambda}^T \mathbf{G}_\mathbf{p})\,\mathrm{d}t \tag{39}$$

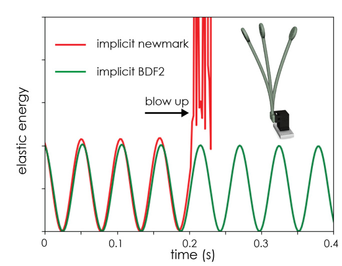
where the *adjoint variables* $\boldsymbol{\lambda}(t)$ are computed by solving the *linear adjoint DAE*

$$\begin{bmatrix} \mathbf{E} & & \\ & \mathbf{M}^T & \\ & & \end{bmatrix} \dot{\boldsymbol{\lambda}} = \left(-\begin{bmatrix} & & \\ & \dot{\mathbf{U}}^T \mathbf{M}_\mathbf{U}^T & \\ & & \end{bmatrix} + \begin{bmatrix} \mathbf{G}_\mathbf{U}^T \\ \mathbf{G}_\mathbf{V}^T \\ \mathbf{G}_\boldsymbol{\Lambda}^T \end{bmatrix}\right)\boldsymbol{\lambda} + \begin{bmatrix} g_\mathbf{U}^T \\ g_\mathbf{V}^T \\ \end{bmatrix}$$

with *initial conditions*

$$\boldsymbol{\lambda}(T) = 0. \tag{40}$$

3 RESULTS



Time Integration. We experimented with explicit 4th-order Runge-Kutta (RK4), implicit Newmark, and implicit second-order backward Euler for simulating our nonlinear and stiff materials, as we illustrate in the inset. Compared to explicit RK4, which blows up immediately even if we set the time step to a $10,000\times$ smaller value, both implicit integrators behave more stably. However, when using the simulation timestep we use to generate our results, implicit Newmark still blows up while implict BDF2 simulates stably and introduces negligible numerical damping.

Motor Profiles and Error Visualization. In Figs. 1, 2, 3, 4 and 5, we show the input and optimized motor profiles for our demonstrations, as well as error plots. We note that with the optimized motor profiles, we can efficaciously suppress visible vibrations in *all* our demonstrations. We also note that there is no clear pattern in the adjustment of motor profiles to minimize vibrations, suggesting that it would be nearly impossible to achieve similar results by manually tuning the motor profiles.

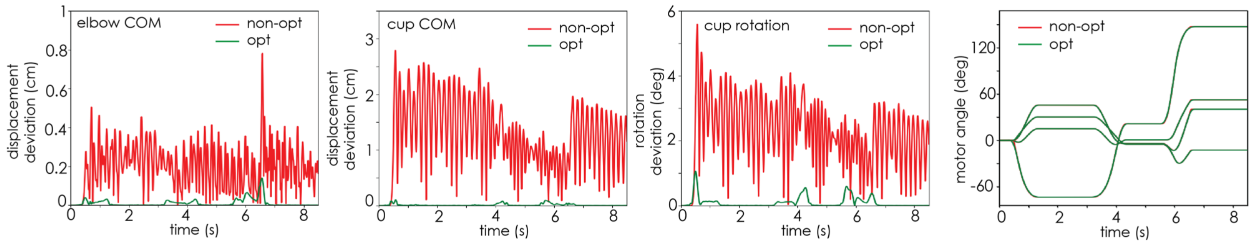


Fig. 1. **Bartender error plots and motor profiles.** We show the position errors at the elbow and the end effector, i.e., cup, as well as the rotation error for the cup. Minor changes to the motor signal suffice to suppress visible vibrations (right).

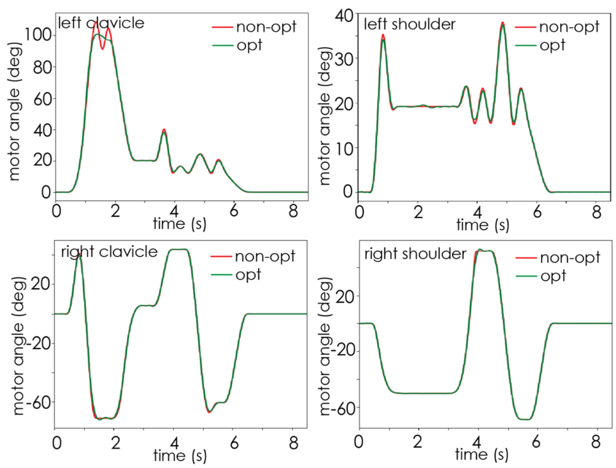


Fig. 2. **Dancing robot motor profiles.** We plot the non-optimized and optimized (with uniform weights) motor profiles for our 4-DOF dancing character. As we can observe from the top left figure, the optimization smoothens the waving motion which adds significant energy to the system.

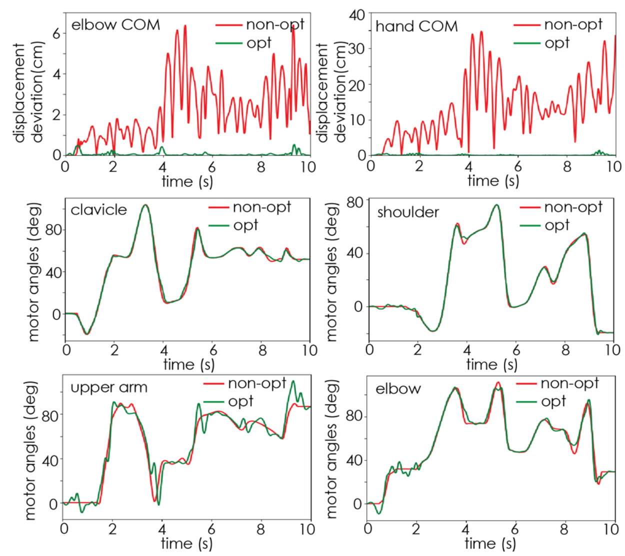


Fig. 3. **Rapper arm error plots and motor profiles.** The first row shows difference in target and simulated trajectories for the elbow and the hand. With our optimized motor profiles (bottom two rows), we suppress the vibration from 35 cm deviation to < 1 cm. Note that the upper arm motor and the elbow motor are coupled in a 4-bar linkage and are actively canceling the vibration via adding higher-frequency motor profiles.

REFERENCES

Yang Cao, Shengtai Li, Linda Petzold, and Radu Serban. 2003. Adjoint sensitivity analysis for differential-algebraic equations: The adjoint DAE system and its numerical solution. *SIAM Journal on Scientific Computing* 24, 3 (2003), 1076–1089.

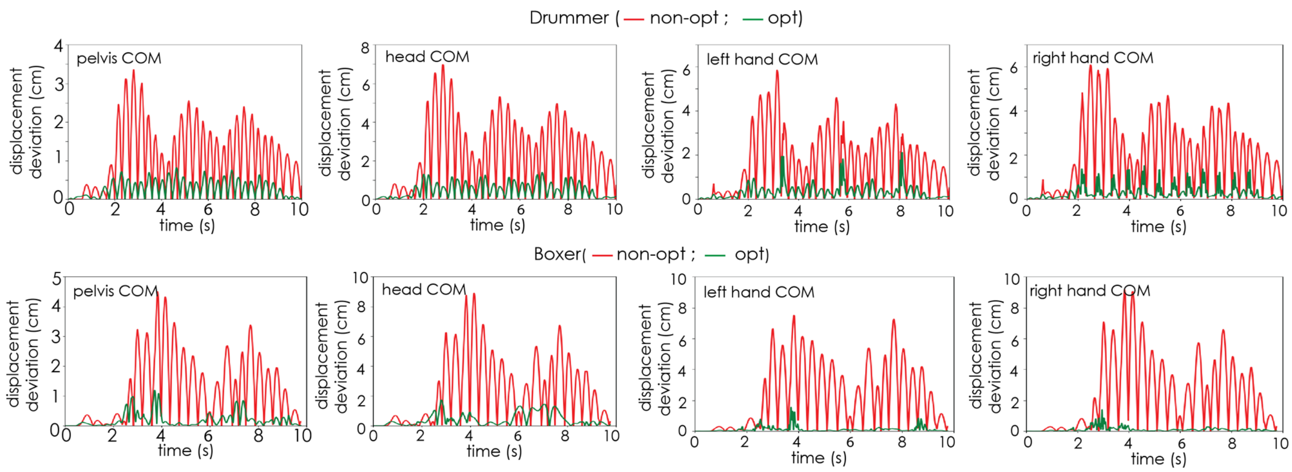


Fig. 4. **Drummer and Boxer error plots.** We show the error plots for our Drummer (top row) and Boxer (bottom row), respectively. Our optimization results significantly reduce deviations from the target.

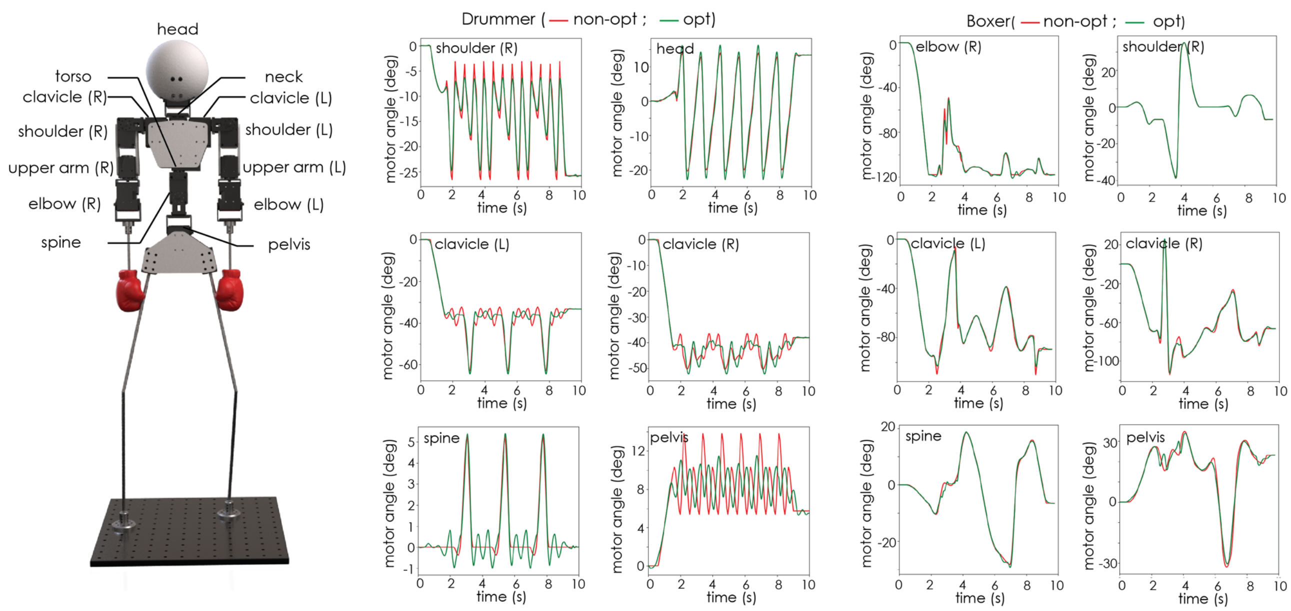


Fig. 5. **Drummer and Boxer motor profiles.** Our 13-DOF full-body character is shown on the left. Motor profiles of 6 representative motors for the drumming (plots in first two columns) and boxing (plots in third and forth columns) animations are shown on the right. Vibrations are suppressed by smoothing some of the motor signals (such as the shoulder (R) and pelvis for drumming) while compensating with motion from other motors (such as head and spine motor for the drumming sequence). Some of them are exactly identical to the input, e.g., the shoulder (R) for boxing.